27.11.25

Reseaux spectraux, exponentiels, et non-abelianisation

0. Intro

I. Supersymmetry, Seiberg-Witten theory, and class S

Among all 4d QFTs, supersymmetric ones are particularly well-behaved:

- often admit <u>manifolds</u> of vacua (of > 0 dimension)
- renormalization is under better control

(Lagrangian density splits into D-terms and F-terms
Perturbative corrections can only contribute to D-terms)

$\Rightarrow$ non-renormalization theorems

$\rightsquigarrow$ <u>holomorphy</u> (Seiberg ~ 90')

The amount of SUSY is encoded as:

$\mathcal{N} = 0, 1, 2, 3, 4, \ldots \rightsquigarrow$ particles of spin > 1.

The higher $\mathcal{N}$, the more rigid the theory:
e.g. $\mathcal{N}=4$ SYM: no renormalization whatsoever.
(minimal $\mathcal{N}=4$ completion of G-YM theory,
G compact reductive Lie groups).
Of course, some interesting physics only occurs for low $\mathcal{N}$
e.g., $\exists$ χ-symmetry $\Rightarrow$ $\mathcal{N}\leq 1$.

<u>EM / S-duality</u>: • Maxwell theory in the vacuum is invariant under $(\vec{E},\vec{B}) \longmapsto (\vec{B},-\vec{E})$.
• Study of <u>monopoles</u> and <u>dyons</u> (Georgi-Glashow) led to
<u>Montonen – Olive conjecture</u>:
G-YM $\xleftrightarrow{\text{dual}}$ G^L-YM

⇝ Works better w/ $\mathcal{N}=4$ SUSY
$\mathcal{N}=4$ G-SYM $\xleftrightarrow{\text{S-duality}}$ $\mathcal{N}=4$ G^L-SYM.
In order to approach the original YM setup, study cases w/ smaller $\mathcal{N}$. <u>Interesting</u>: $\mathcal{N}=2$: Seiberg-Witten theory. ('94)
• There is a notion of S-duality for 4d $\mathcal{N}=2$ theories.
• On the <u>Coulomb branch</u> of the moduli space of SUSY vacua of any 4d $\mathcal{N}=2$ theory (where the low-energy effective dynamics is a 4d $\mathcal{N}=2$ generalized Maxwell theory), the physics defines, and is encoded in, a complex completely integrable system.
<u>Known examples</u> (~'95, '96) are special cases of Hitchin integrable systems. Pure $\mathcal{N}=2$ SYM $\leftrightarrow$ Toda.
$\mathcal{N}=2^*$ SYM $\leftrightarrow$ CM.
<u>Fruitful correspondence</u> (global structure: 2504.01502)
CM for <u>Lie groups</u> (not algebras)
<u>More recently</u>: $\forall$ Hitchin system, $\exists$ a corresponding 4d $\mathcal{N}=2$ theory

→ class S (Gaiotto '08, Gaiotto-Moore-Neitzke '09)

The spectral curve of the HS is the SW curve of the theory

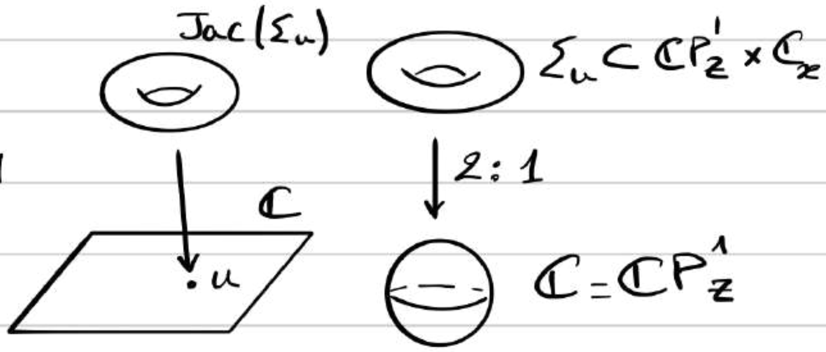
λ Σ Seiberg-Witten curve

(meromorphic SW differential)

↓

C UV curve

$\boxed{\Rightarrow \text{Geometric study of the CB of 4d } \mathcal{N}=2 \text{ theories.}}$

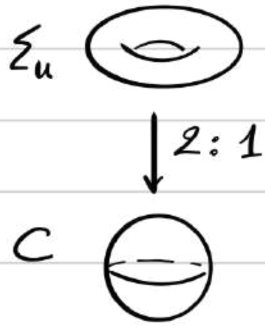


Example: 4d $\mathcal{N}=2$ $SU(2)$ SYM

Σ_u: $\Lambda^2/z + \Lambda^2 z = x^2 - u$

$\lambda = x \dfrac{dz}{z}$

II. Spectral networks and flat G-connections on surfaces.



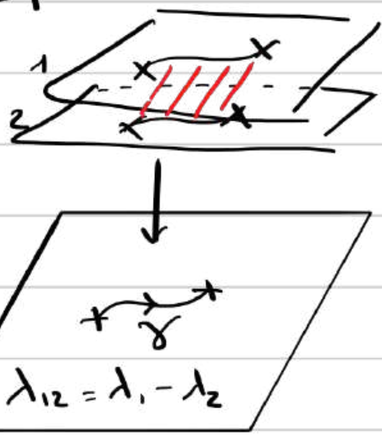
BPS state: "short" irrep of the supersymmetry algebra

⇒ Remarkable stability properties (central for S-duality)

Every state in a 4d $\mathcal{N}=2$ theory satisfies an ineq.:

mass $M \geq |Z|$ $\quad Z \in \mathbb{C}$ central charge.

BPS states saturate it: $M = |Z|$



⇝ Can be described as finite strings on C.

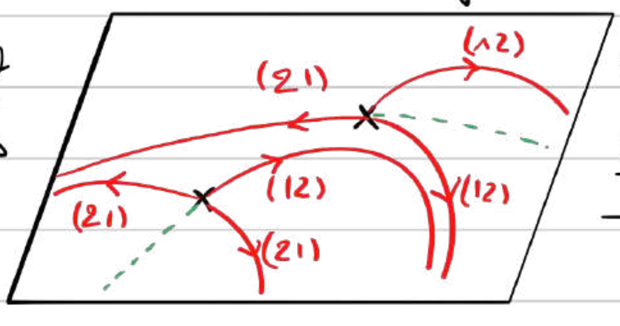
Locally: $\quad M_\gamma = \int_\gamma |\lambda_{12}|; \quad Z_\gamma = \int_\gamma \lambda_{12}$

BPS diff eq: $\quad \lambda_{12}(\dot{\gamma}(t)) = e^{i\vartheta} \quad$ $\vartheta \in [0, 2\pi)$ fixed

Fixed u, fixed ϑ

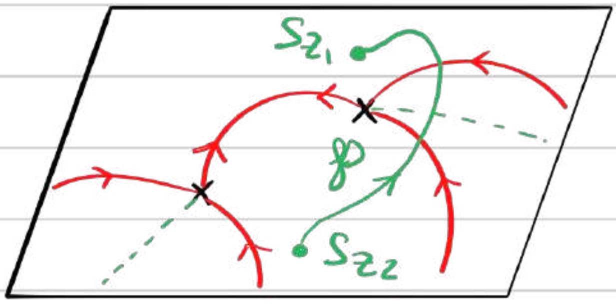
Keep only the curves starting at the BP.

→ Spectral Network



Vary ϑ: the SN jumps → BPS states.

Spectral networks (GMN '13) are defined more rigorously via the study of 4d $\mathcal{N}=2$ theories with supersymmetric surface defects S_{z_1}, S_{z_2} for $z_1, z_2 \in C$, and SUSY interfaces between S_{z_1} & S_{z_2}, denoted by $L_\wp$.



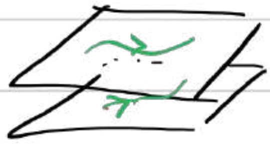
C

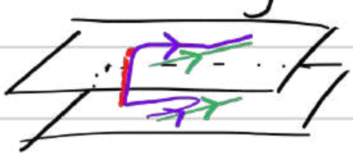
On general grounds, the physics of $L_\wp$ depends only on the homology/homotopy class $[\wp]$ of $\wp$.

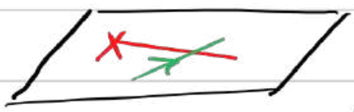
In fact, the spectral networks assigns to each such $\wp$ a formal sum of paths on Σ, in a homotopy invariant way, i.e. a formal sum of homot. classes on Σ to each homot. class on C. Call this map PL.

Thus, if ∇^{ab} is a flat abelian ($\mathbb{C}^*$) connection on Σ, i.e. $\pi_1(\Sigma) \longrightarrow \mathbb{C}^*$ (parallel transport) one obtains by pullback a non-abelian ($SL_2(\mathbb{C})$ here) flat connection ∇^{na} on C. This is non-abelianization (cf. NAHC for Higgs bundles).

This works locally, via detour rules:



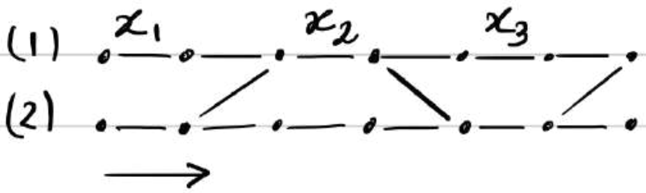




$\Rightarrow$ Formally similar to the parametrizat° of Double Bruhat cells in SL_2 / PGL_2:

Example: $M_\gamma = \begin{pmatrix} x_1 & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_2 & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} x_3 & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}$

$$M_\gamma = \begin{pmatrix} x_1x_2 + x_1x_2x_3 & x_1x_2 \\ 1 + x_2 + x_2x_3 & 1 + x_2 \end{pmatrix}$$



Planar network

Bruhat decomposition : $G = \coprod_{w \in W} BwB$

Double Bruhat cell : $\forall (u,v) \in W$, $G^{u,v} := (BuB) \cap (B_- v B_-)$

$\rightsquigarrow$ links with total positivity, cluster algebras/varieties, integrable systems in Poisson-Lie groups.

Jump of the network $\Leftrightarrow$ cluster mutation.

III. Exponential Networks, cluster integrable systems, and affine Double Bruhat Cells.

5d $\mathcal{N}=1$ theory on $\mathbb{R}^{1,4} \times S^1 \rightsquigarrow$ effective 4d $\mathcal{N}=2$ theory

Nekrasov ('97): the corresponding SW int. system is relativistic.

5d $\mathcal{N}=1$ SCFTs have mainly been studied through string theoretic techniques (YM theories are IR free in $d \geq 5$, i.e. non-renormalizable; however, $\exists$ non-trivial IR fixed points)

Geometric engineering M-theory on $X \times S^1$
$\parallel$ 5d $\mathcal{N}=1$ SCFT

X non-compact toric affine CY3 singularity (conifold, local F_0)

Type IIA on $X \xleftrightarrow[\text{Symmetry}]{\text{Mirror}}$ Type IIB on $\tilde{X}$

$\tilde{X} = \{uv = P(x,z)\} \subset \mathbb{C}^2_{u,v} \times (\mathbb{C}^*)^2_{x,z}$

$\text{Mirror } \Sigma = \{P(x,z)=0\} \subset (\mathbb{C}^*)_{x,z}$ e.g. $\{\frac{\alpha}{z} + \frac{x}{z} + x + x^2 = 0\}$

$\downarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow 2:1$

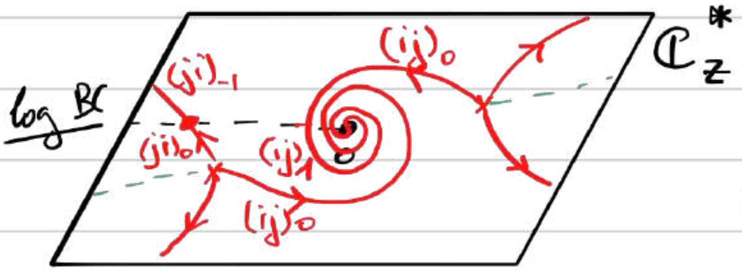
$\mathbb{C}^*_z$ $\qquad\qquad\qquad\qquad$ (conifold) $\qquad\qquad$ $\mathbb{C}^*_z$

Construction of EN: Similar to SN, except that now $\lambda = \log x \frac{dz}{z}$

$\log x$ is multivalued $\log x \sim \log x + 2i\pi k$

Curves now depend on the determination of the log:

$$(\log x_i - \log x_j + 2i\pi k)\frac{dz}{z}(\dot{\gamma}(t)) = e^{i\vartheta} \text{ fixed}$$



Jumps of EN as ϑ varies:

BPS states of 5d $\mathcal{N}=1$ theory

(DT invariants of X).

Just as for SN, one can define a non-abelianization procedure for EN via detour rules.

i.e. $\exists$ a homotopy invariant path-lifting map PL_ξ

$\pi_1(\mathbb{C}^\times) \longrightarrow$ formal sum of hom. classes of paths in Σ.

Question: What is $PL_\xi^*(\nabla^{ab})$?

Claim: It is an element of a specific double Bruhat cell (determined by X) of the affine Poisson–Lie group $\widehat{PGL_N}(\mathbb{C})$ (N is also determined by X).

In fact the DBC are phase spaces of (cluster) integrable systems, which are the relat. int. systems corresponding to the theory (Fock–Marshakov '14)

Rmks/generalization: 1) Cluster int systems & DT invariants

2) Links with resurgence, quantum mirror curves, difference equations.