25.11.2025

Double Bruhat cells and exponential networks

<u>Intro</u>: <u>Goal</u>: Explain the geometric essence of the non-abelianization procedure that exponential networks define.

Spectral networks generalize the <u>cluster charts</u> of FG on specific moduli spaces of (na) flat connections on S.

$$\Sigma \xrightarrow{\text{ramified covering}} S$$

(homotopy invariant path lifting + pull-back of abelian connection)

- EN generalize SN and also provide a HIPL map. $\boxed{\text{Charts on what?}}$

<u>I. Double Bruhat Cells</u>

Fomin-Zelevinsky '98 (9802056) G simply-connected $\mathbb{C}$ simple
<u>Fock-Goncharov</u> '05 (0508408) G adjoint $\mathbb{C}$ semi-simple
<u>Fock-Marshakov</u> '14 (1401.1606) G adjoint $\mathbb{C}$ affine (coextended)

Let $\mathfrak{g}$ = f.d. simple complex Lie algebra of rank r.
Chevalley generators $\{e_i, h_i, e_{\bar{i}}\}$, $i \in \Pi$, $\bar{i} \in \overline{\Pi}$

e.g. $[h_i, h_j] = 0$, $[h_i, e_j] = \delta_{ij} e_j$

$[h_i, e_{\bar{j}}] = -\delta_{ij} e_{\bar{j}}$, $[e_i, e_{\bar{i}}] = \sum_{j \in \Pi} C_{ij} h_j$,

<u>Serre relations</u>.

Let G be the adjoint complex Lie group s.t. $\mathfrak{g} = \mathrm{Lie}(G)$.

H Cartan, B, B_- opposite Borels, W Weyl group

Bruhat decompositions $G = \coprod_{w \in W} BwB = \coprod_{\bar{w} \in W} B_- \bar{w} B_-$

$\forall (u,v) \in W \times W$:

$$\boxed{G^{u,v} := (BuB) \cap (B_- v B_-)}$$

$\rightarrow$ many links with total positivity.

<u>Remark:</u> $\dim(G^{u,v}) = r + \ell(u) + \ell(v)$,
$\dim(G^{u,v}/\mathrm{Ad}\,H) = \ell(u) + \ell(v)$.

We are interested in the <u>cluster parameterization</u> of these cells. <u>Generators of G:</u>
$\forall i \in \Pi, \bar{\imath} \in \overline{\Pi}$, $E_i = \exp(e_i)$, $E_{\bar{\imath}} = \exp(e_{\bar{\imath}})$,
$\forall x \in \mathbb{C}^\times$, $H_i(x) = \exp(h_i \log x)$.

$m = \ell(u) + \ell(v)$

$\forall$ <u>double reduced word</u> $\underline{i} = (s_{i_1} \dots s_{i_m})$ of (u,v), where $\forall k$, $i_k \in \Pi \cup \overline{\Pi}$, the map

$$\varphi_{\underline{i}} : (\mathbb{C}^\times)^{r+m} \longrightarrow G$$
$$x_1, \dots, x_{r+m} \longmapsto H_{i_1}(x_1)E_{i_1} \cdots H_{i_m}(x_m)E_{i_m} H_1(x_{m+1}) \cdots H_r(x_{m+r})$$

is a birational isomorphism between $(\mathbb{C}^\times)^{r+m}$ and $G^{u,v}$.
(If $\underline{i}$ is not reduced, this is not an embedding anymore)
<u>Similarly:</u> $\varphi_{\underline{i}} : (\mathbb{C}^\times)^{r} \overset{\sim}{\dashrightarrow} G^{u,v}/\mathrm{Ad}\,H$

$$x_1, \dots, x_r \longmapsto H_{i_1}(x_1)E_{i_1} \cdots H_{i_m}(x_m)E_{i_m}$$

<u>Example:</u> $G = PGL_2$ $\quad H = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$, $E_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $E_{\bar{1}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

<u>$u = v = s$</u> $\quad \varphi_{s\bar{s}}(x_1, x_2, x_3) = \begin{pmatrix} x_1x_2x_3 + x_1x_3 & x_1 \\ x_3 & 1 \end{pmatrix}$

$$\varphi_{\bar{s}s}(y_1, y_2, y_3) = \begin{pmatrix} y_1y_2y_3 & y_1y_2 \\ y_2y_3 & y_2 + 1 \end{pmatrix}$$

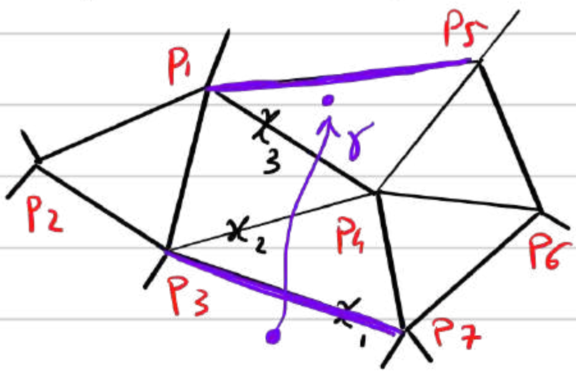
<u>Note:</u> $\varphi_{s\bar{s}}(x_1, x_2, x_3) = \varphi_{\bar{s}s}(x_1(1+x_2), x_2^{-1}, x_3(1+x_2))$
(cluster $\mathcal{X}$-mutations)
$\Rightarrow$ For every double reduced word $\underline{i}$ for $(u,v) \in W \times W$, we obtain a <u>cluster chart</u> on $G^{u,v}$.

II. $PGL_2(\mathbb{C})$-flat connections on surfaces and non-abelianization

Consider the moduli space $\mathcal{X}_{PGL_2,S}(\mathbb{C})$ of framed flat $PGL_2(\mathbb{C})$ connections on a punctured surface $S = S_{g,n}$ (of hyp. type)

framed = choice of a framing at each puncture p
= choice of an eigenline of the monodromy at p

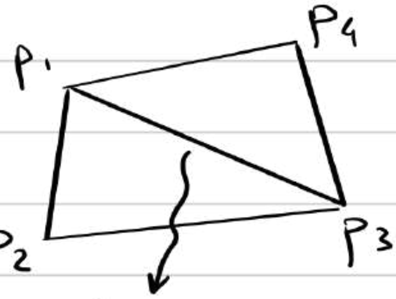
Ideal triangulation of $S_{g,n}$: triang. w/ vertices at the punctures



Piece of the triangulation

Framing $p_i \in \mathbb{P}^1(\mathbb{C})$ ($\pi_1(S)$-equiv collection)

FG coordinates '03 (0311149)



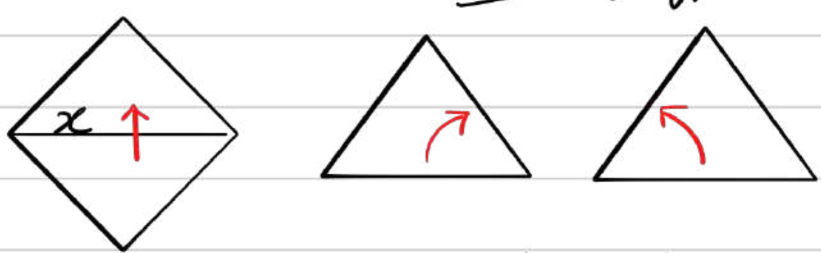
- Well-defined (π_1-invariant)

$$(\mathbb{C}^\times)^{E(\Gamma)} \overset{\sim}{\dashrightarrow} \mathcal{X}_{PGL_2,S}(\mathbb{C})$$

$\mathcal{X}(p_1, p_2, p_3, p_4)$

- $\mathcal{X}_{PGL_2,S}(\mathbb{C})$ is a cluster variety of type $\mathcal{X}$.

Reconstruction:

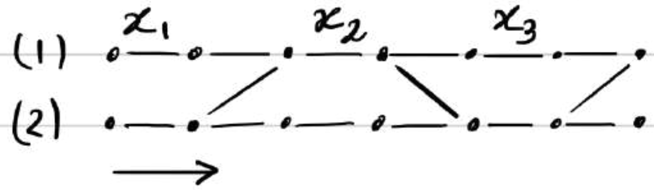


$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Example: $M_\gamma = \begin{pmatrix} x_1 & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_2 & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} x_3 & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}$

$$M_\gamma = \begin{pmatrix} x_1x_2 + x_1x_2x_3 & x_1x_2 \\ 1+x_2+x_2x_3 & 1+x_2 \end{pmatrix}$$

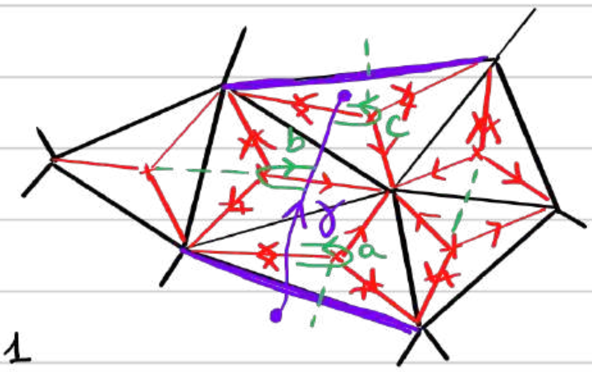
(Hollands-Neitzke '13
1312.2979)

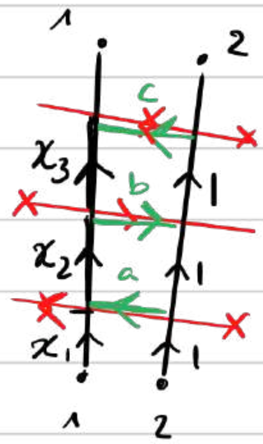


<u>Planar network</u>

This is formally similar to the coordinates on $\mathcal{X}_{PGL_2,S}(\mathbb{C})$ obtained from the non-abelianization that spectral networks define!

$\mathcal{W}_\Gamma$





$\Sigma \xrightarrow{2:1} S$

($\exists$ a combinatorial construction of Σ).

$$[\gamma] \longrightarrow \overset{x_1x_2x_3}{[\gamma^{(11)}]} + \overset{x_1x_2}{[\gamma^{(11)}_{bc}]} + \underset{x_1x_2}{[\gamma^{(12)}_{b}]} + \underset{x_2x_3}{[\gamma^{(21)}_{a}]}$$
$$+ \underset{1}{[\gamma^{(21)}_{c}]} + \underset{x_2}{[\gamma^{(21)}_{abc}]} + \underset{1}{[\gamma^{(22)}]} + \underset{x_2}{[\gamma^{(22)}_{ab}]}$$

This map is the <u>homotopy-invariant path-lifting</u>

$$PL^{\mathcal{W}}_{\chi} : \mathbb{Z}[\mathcal{P}(S)] \longrightarrow \mathbb{Z}^{tw}_{R}[\mathcal{P}(\Sigma)]$$

(cf. 2412.03588)

(<u>Rmk</u>: There also exists $PL^{\mathcal{W}}_{\mathcal{A}} : \mathbb{Z}[\widetilde{\mathcal{P}}(S)] \longrightarrow \mathbb{Z}[\widetilde{\mathcal{P}}(\Sigma)]$).

Here $\mathcal{W}_\Gamma$ is a very simple SN (WKB spectral network) but these last map exist for any SN $\mathcal{W}$.

$\Rightarrow$ non-abelianization : $\underline{\nabla^{na} = (PL^{\mathcal{W}})^* \nabla^{ab}}$

- BPS states in 4d $\mathcal{N}=2$ theories in class S : cluster transformations

III. Exponential networks and affine double Bruhat cells

EN generalize SN to 5d $\mathcal{N}=1$ theories (on S^1) obtained from <u>geometric engineering</u>: M-theory on X, where X is a (resolved) toric affine CY3.

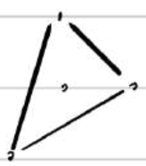
BPS states in these theories $\longleftrightarrow$ DT invariants of X.

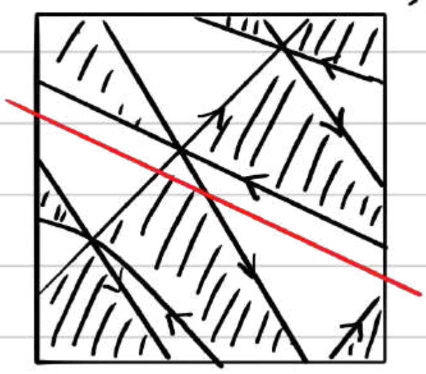
EN are subordinate to a ramified covering $\begin{array}{c}\Sigma\\ \downarrow\\ \mathbb{C}^\times\end{array}$, where $\Sigma \subset (\mathbb{C}^\times)^2$ is the zero locus of some Laurent polynomial ("mirror curve" of X).

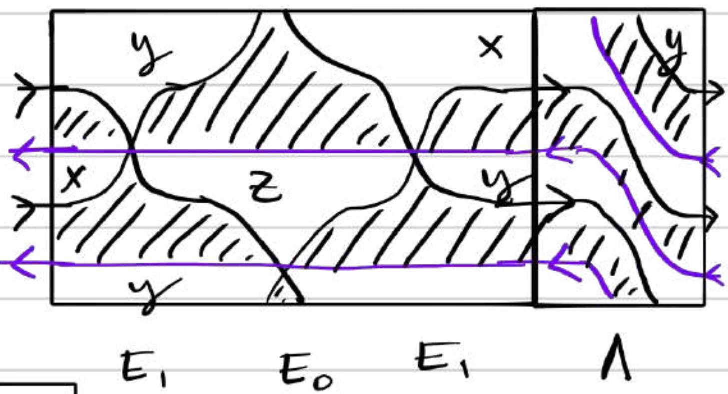
EN also define homotopy invariant path-lifting maps $PL^{\mathcal{E}}_{\mathcal{A}}, PL^{\mathcal{E}}_{\mathcal{X}}$.
(Banerjee, Longhi, Romo '18 1811.02875)

If ∇^{ab} is an abelian connection on Σ, what kind of objects are $PL^{\mathcal{E}\,*}_{\mathcal{A}}(\nabla^{ab})$ and $PL^{\mathcal{E}\,*}_{\mathcal{X}}(\nabla^{ab})$?

<u>Claim</u>: They are <u>double Bruhat cells</u> in affine PL groups.
(Fock–Marshakov '14, 1401.1606 for type $\mathcal{X}$).







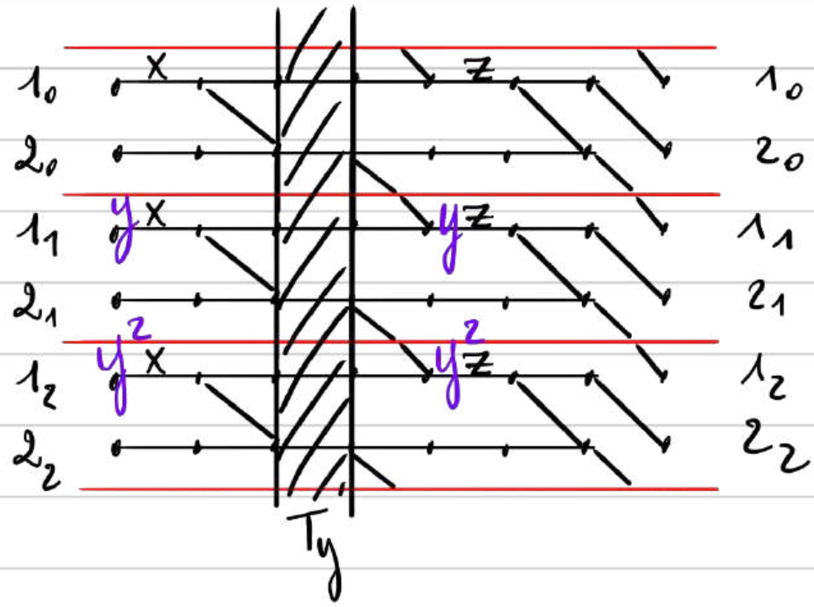
$$\boxed{xyz = 1}$$

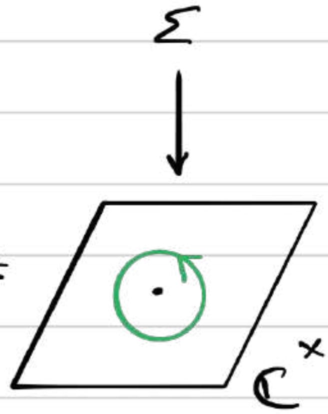
$$T_y = y^{\lambda\partial/\partial\lambda}$$
$$T_y(\lambda^k) = (y\lambda)^k$$

$$H_1(x)\, E_1\, H_0(y)\, E_0\, H_1(z)\, E_1\, \Lambda$$

$$= \begin{pmatrix} x & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} T_y \begin{pmatrix} 1 & \\ \lambda & 1 \end{pmatrix}\begin{pmatrix} z & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (xzy + xy)\lambda + y^2 xz\lambda^2 & xz + xyz\lambda \\ y\lambda + zy^2\lambda^2 & zy\lambda \end{pmatrix} T_y$$





Remarks: 1) We obtain one element $g \in \hat{G}^{\#}$

$\Leftrightarrow$ affine na flat connection on $\mathbb{C}^\times$.

Generalization ?

2) We can construct (very) simple EN on $\mathbb{C}^\times$ for each reduced word for $(u,v) \in \hat{W} \times \hat{W}$, for which na & FM coordinates coincide.

$\Rightarrow$ we understand na for all EN which can be obtained from those via homotopies

3) These cluster integrable systems are the SW integrable systems for 5d $\mathcal{N}=1$ theories on S^1 (Nekrasov '96 9609219).

Cluster dualities ?

4) Simple models for DT invariants ?

5) Link with resurgence, quantum mirror curve, difference equations.